

ON THE DUNKL INTERTWINING OPERATOR

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ABSTRACT. Dunkl operators are differential-difference operators parametrized by a finite reflection group and a weight function. The commutative algebra generated by these operators generalizes the algebra of standard differential operators and intertwines with this latter by the so-called intertwining operator. In this paper, we give an integral representation for the operator $V_k \circ e^{\Delta/2}$ for an arbitrary Weyl group and a large class of regular weights k containing those of non negative real parts. Our representing measures are absolute continuous with respect the Lebesgue measure in \mathbb{R}^d , which allows us to derive out new results about the intertwining operator V_k and the Dunkl kernel E_k . We show in particular that the operator $V_k \circ e^{\Delta/2}$ extends uniquely as a bounded operator to a large class of functions which are not necessarily differentiables. In the case of non negative weights, this operator is shown to be positivity-preserving.

1. INTRODUCTION AND PRELIMINARIES

Dunkl operators are first-order differential-difference operators associated to a finite reflection group and a weight function. They form a commutative algebra which generalizes the algebra of standard differential operators. Nowadays, there is a harmonic analysis associated to these operators which extends the euclidean Fourier analysis by using the action of a Weyl group generated by some root systems in a finite dimensional vector space. During the last years, this theory has attracted considerable interests in various fields of mathematics[5, 13] and also in physics[7].

In this theory, the so-called Dunkl kernel, plays the primordial role of the “exponential” and is obtained as the image of the standard exponential by another key operator referred to as the intertwining operator, named after its action of intertwining the Dunkl differential operators algebra and the algebra of classical differential operators.

One challenging problem in the Dunkl operators theory is to find a representing measures for the action of the intertwining operator on the space of polynomials. This problem is solved by Rösler[12] in the case of non negative weights and the general case remains so far an open problem.

In this paper we reduce the gap and get closer to these representing measures. We give a family of representing measures of the action of the operator $V_k \circ e^{\Delta/2}$ on the space of polynomials for an arbitrary Weyl group and a large class of regular weights k containing those of non negative real parts. Our representing measures are absolute continuous with respect to the Lebesgue measure in \mathbb{R}^d , and reveal new results about the intertwining operator V_k and the Dunkl kernel E_k .

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We show that the operator $V_k \circ e^{\Delta/2}$ extends uniquely as a bounded operator to a large class of functions, including polynomials, which are not necessarily differentiable. In the case when k is non negative, this operator is shown to be positive-preserving.

This paper contains two main results and is organized as follows. The current section serves to introduce the Dunkl operators and the common notations used throughout this paper. In section 2 we gather all technical results requested by the developments in section 3 where our main results are presented. The first one gives an integral representation for the operator $V_k \circ e^{\Delta/2}$. The second one extends the operator $V_k \circ e^{\Delta/2}$ as bounded operator to a large class of functions containing polynomials, and show its positivity-preserving property when the weight k is non negative.

We consider the euclidean space \mathbb{R}^d equipped with its canonical inner product $\langle \cdot, \cdot \rangle$ which we extend as a bilinear form on $\mathbb{C}^d \times \mathbb{C}^d$ again denoted by $\langle \cdot, \cdot \rangle$. For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, we let

$$\|z\|^2 := z_1^2 + \dots + z_d^2.$$

Fix a root system R and consider the finite group G generated by the reflections σ_α where $\alpha \in R$ and

$$\sigma_\alpha(x) = x - 2 \langle x, \alpha \rangle \alpha / \|\alpha\|^2, \quad (x \in \mathbb{R}^d).$$

Let $\mathcal{P} := \mathbb{C}[\mathbb{R}^d]$ denotes the \mathbb{C} -algebra of polynomial functions on \mathbb{R}^d and \mathcal{P}_n , $n \in \mathbb{N}$, its subspace consisting of all homogeneous polynomials of degree n , and for $p \in \mathcal{P}_n$, for some $n \geq 0$, we set

$$\|p\|_{\mathbb{S}} := \sup_{\|x\|=1} |p(x)|.$$

The action of G on functions is defined by

$$(L_g \cdot f)(x) := f(g \cdot x), \quad x \in \mathbb{R}^d$$

Set $R_+ := \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$ for some $\beta \in \mathbb{R}^d$ such that $\langle \alpha, \beta \rangle \neq 0$ for all $\alpha \in R$.

Let k be a parameter function on R , that is, $k : R \rightarrow \mathbb{C}$ and G -invariant.

For $\xi \in \mathbb{R}^n$, the Dunkl operator T_ξ on \mathbb{R}^d associated to the group G and the parameter function k is given by

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad x \in \mathbb{R}^d$$

where $f \in C^1(\mathbb{R}^d)$. In the sequel we will write T_j in place of $T_{e_j}(k)$, where e_j is a vector from the standard basis of \mathbb{R}^d , $j = 1, \dots, d$.

Consider the vector space M of all parameter functions and let

$$M^{reg} := \{k \in M : \cap_{\xi \in \mathbb{R}^d} \ker(T_\xi(k)) = \mathbb{C} \cdot 1\}$$

be the set of regular parameter functions.

An important result in Dunkl theory, established in [6], states that $k \in M^{reg}$ if and only if there exists a unique isomorphism V_k of \mathcal{P} satisfying

$$V_k(\mathcal{P}_n) \subset \mathcal{P}_n, \quad V_k(1) = 1 \quad \text{and} \quad T_\xi V_k = V_k \partial_\xi, \quad \forall \xi \in \mathbb{R}^d. \quad (1)$$

The operator V_k is called the intertwining operator.

In [9] a construction of the intertwining operator was given for a large class of regular weights. We recall here some notations and some key results from [9].

Consider the operator $A := A_k$ defined on functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ by

$$A(f) := \sum_{\alpha \in R_+} k(\alpha) L_{\sigma_\alpha} \cdot f. \quad (2)$$

It is clear that for all integer $n \geq 0$, the space \mathcal{P}_n is invariant under the action of A . For $n \geq 1$, we set $A_n := A|_{\mathcal{P}_n}$, we define

$$\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha), \quad (3)$$

and we consider the operator $W_n := W_{n,k}$ defined in \mathcal{P}_n by

$$W_n = (n + \gamma) - A_n.$$

A straightforward calculations show that W_n is a Euler-type operator satisfying

$$W_n(p)(x) = \sum_{j=1}^d x_j T_j(p)(x),$$

for all polynomial $p \in \mathcal{P}_n$ and all $x \in \mathbb{R}^d$.

We denote by M^* the set of all parameter functions k for which W_n is invertible for all $n \geq 1$. As mentioned in [9] (see also [10, Corollary 2.3]), $M^* \subset M^{reg}$, and contains a large class of parameter functions including those of non negative real part. For $k \in M^*$, we set

$$H_n = H_{n,k} := ((n + \gamma) - A_n)^{-1}.$$

One key result from [9] needed in this paper is the following.

Theorem 1.1 ([9]). *Assume that $k \in M^*$. Then there exists a sequence of functions $\lambda_n := \lambda_{n,k} : G \rightarrow \mathbb{C}$, $n \geq 1$, such that for all $n \geq 1$ we have $H_n = \sum_{g \in G} \lambda_n(g) L_g$ and*

$$V_k(p)(x) = \sum_{g_1, \dots, g_n \in G} \left(\prod_{i=1}^n \lambda_n(g_i) \right) \partial_{g_1 \dots g_n x} \dots \partial_{g_{n-1} g_n x} \partial_{g_n x} p$$

for all $p \in \mathcal{P}_n$ and $x \in \mathbb{R}^d$. Moreover, there exists a positive constant $\delta := \delta(k)$ such that

$$|\lambda_n(g)| \leq \frac{\delta}{n}$$

for all $n \geq 1$ and all $g \in G$.

From now on, we assume that $k \in M^*$, unless otherwise mentioned, and we refer to λ_n , $n \geq 1$, and the constant δ as they were introduced in Theorem 1.1.

Remark 1.2. *When k is real valued, then W_n and H_n lie together in the algebra $\left\{ \sum_{g \in G} c(g) L_g, c(g) \in \mathbb{R} \right\}$. As a consequence, by use of Theorem 1.1 we see that V_k is real valued on polynomials $p \in \mathbb{R}[\mathbb{R}^d]$.*

The next useful estimate for V_k , valid in the case $k \in M^*$, is derived out from Theorem 1.1:

Proposition 1.3. *Let $n \geq 1$. Then for all $p \in \mathcal{P}_n$ and $x \in \mathbb{R}^d$ we have*

$$|V_k(p)(x)| \leq \frac{(\delta |G| \|x\|)^n}{n!} \|p\|_{\mathbb{S}}.$$

The Dunkl kernel $(x, y) \mapsto E_k(x, y)$, is a fundamental tool in the harmonic analysis associated to Dunkl operator theory. Indeed, it plays the role of a generalized exponential function and it is used to define the Dunkl Fourier transform.

For all $y \in \mathbb{R}^d$, the function $f : x \mapsto E_k(x, y)$ is the unique solution of the system

$$f(0) = 1, \quad (T_\xi f)(x) = \langle \xi, y \rangle f(x)$$

for all $\xi \in \mathbb{R}^d$. Moreover, the map $(k, x, y) \mapsto E_k(x, y)$ is holomorphic in the set $M^{\text{reg}} \times \mathbb{C}^d \times \mathbb{C}^d$.

We list here some of its basic properties needed for the sequel. For more details on this kernel see [3, 1] and the references there in.

Proposition 1.4. ([1]) *Let $x, y \in \mathbb{C}^d$, $\lambda \in \mathbb{C}$ and $g \in G$. Then*

- (1) $E_k(x, 0) = 1$,
- (2) $E_k(x, y) = E_k(y, x)$,
- (3) $E_k(\lambda x, y) = E_k(x, \lambda y)$,
- (4) $E_k(gx, y) = E_k(x, g^{-1}y)$.

In [9], the kernel E_k is obtained in the case $k \in M^*$ by

$$E_k(x, y) = V_k \left(e^{\langle \cdot, y \rangle} \right) (x) = \sum_{n=0}^{\infty} E_n(x, y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{C}^d$$

where $E_n(x, y) := \frac{1}{n!} V_k(\langle \cdot, y \rangle^n)(x)$.

A direct application of Theorem 1.1 gives us

$$E_n(x, y) = \sum_{g_1, \dots, g_n \in G} \left(\prod_{i=1}^n \lambda_n(g_i) \right) \langle g_1 \dots g_n x, y \rangle \dots \langle g_n x, y \rangle \quad (4)$$

for all $n \geq 1$ and all $x, y \in \mathbb{C}^d$.

2. PREPARATORY RESULTS

The main task of this section is to prepare the ground for the results in section 3.

We begin by recalling a useful identity for n -linear symmetric forms in \mathbb{R}^d . Let ϕ be an n -linear form in \mathbb{R}^d . Then following [11, Theorem 9] we have:

$$\sup_{\|z\|=1} |\phi(z, \dots, z)| = \sup_{\|z_1\|=1, \dots, \|z_n\|=1} |\phi(z_1, \dots, z_n)| \quad (5)$$

The next technical Lemma is the corner stone for our paper. Together with (4), lead to a useful estimates for the polynomials $E_n(x, y)$ interesting for themselves and required for our developments in section 3.

Lemma 2.1. *Let n be a positive integer and p the polynomial given by*

$$p(y) = \langle \xi_1, y \rangle \times \dots \times \langle \xi_n, y \rangle$$

where $\xi_i \in \mathbb{R}^d$ such that $\|\xi_1\| = \dots = \|\xi_n\| = c$ for all $i = 1, \dots, n$. Then for all $y \in \mathbb{R}^d$ and $0 \leq m \leq n/2$, we have

$$|\Delta^m p(y)| \leq \frac{n! c^n d^m}{(n-2m)!} \|y\|^{n-2m}$$

where Δ denotes the (standard) laplacian and the differentiation is taken with respect to y .

Proof. Consider the n -linear symmetric form ϕ defined by $p(y) = \phi(y, \dots, y)$. Letting (e_1, \dots, e_d) denotes the standard basis of \mathbb{R}^d , direct calculations give us

$$\Delta p(y) = \Delta \phi(y, \dots, y) = n(n-1) \sum_{i_1=1}^d \phi(e_{i_1}, e_{i_1}, y, \dots, y),$$

and by induction we get

$$\Delta^m p(y) = \frac{n!}{(n-2m)!} \sum_{i_1=1}^d \cdots \sum_{i_m=1}^d \phi(e_{i_1}, e_{i_1}, \dots, e_{i_m}, e_{i_m}, y, \dots, y), \quad (6)$$

whenever $2m \leq n$. Keeping (5) in mind, we get

$$\begin{aligned} |\phi(e_{i_1}, e_{i_1}, \dots, e_{i_m}, e_{i_m}, y, \dots, y)| &\leq \|y\|^{n-2m} \sup_{\|z_1\|=1, \dots, \|z_d\|=1} |\phi(z_1, \dots, z_d)| \\ &\leq \|y\|^{n-2m} \sup_{\|z\|=1} |p(z)| \\ &\leq c^n \|y\|^{n-2m}. \end{aligned}$$

Coming back to (6), we infer that

$$|\Delta^m p(y)| \leq \frac{n!d^m}{(n-2m)!} c^n \|y\|^{n-2m},$$

as claimed. \square

To indicate the relevant variable, we will sometimes use the notations $\Delta^{m,y} E_n(x, y)$, $e^{-\Delta/2, y} E_n(x, y)$ or $(\Delta^{m, \cdot} E_n(x, \cdot))(y)$ and $(e^{-\Delta/2, \cdot} E_n(x, \cdot))(y)$. Sometimes, for the sake of simplicity, we will write only $\Delta^m E_n(x, y)$ and $e^{-\Delta/2} E_n(x, y)$ when the relevant variable of the differentiation is known.

The next proposition is the fundamental tool for our developments in section 3.

Proposition 2.2. (1) *For all nonnegative integer n and $0 \leq m \leq n/2$, we have*

$$|(\Delta^{m, \cdot} E_n(x, \cdot))(y)| \leq \frac{d^m}{(n-2m)!} (\delta |G| \|x\|)^n \|y\|^{n-2m}$$

for all $x, y \in \mathbb{R}^d$.

(2) *The series $\sum_{n=0}^{\infty} (e^{-\Delta/2, \cdot} E_n(x, \cdot))(y)$ converges uniformly with respect to x and y in each bounded subset of $\mathbb{R}^d \times \mathbb{R}^d$. Moreover,*

$$\sum_{n=0}^{\infty} \left| (e^{-\Delta/2, \cdot} E_n(x, \cdot))(y) \right| \leq e^{(\delta \sqrt{d} |G| \|x\|)^2 / 2} e^{\delta |G| \|x\| \|y\|}.$$

Proof. Fix $x \in \mathbb{R}^d$.

(1) Lemma 2.1 applied to each polynomial

$$Q : y \mapsto \langle g_1 \dots g_n x, y \rangle \langle g_2 \dots g_n x, y \rangle \dots \langle g_n x, y \rangle,$$

where $x \in \mathbb{R}^d$ and $g_1, \dots, g_n \in G$ fixed, gives us

$$|\Delta^m Q(y)| \leq \frac{n!d^m}{(n-2m)!} \|x\|^n \|y\|^{n-2m} \quad (7)$$

Appealing to Theorem 1.1 we have

$$\left| \prod_{i=1}^n \lambda_n(g_i) \right| \leq \frac{\delta^n}{n!} \quad (8)$$

for all $g_1, \dots, g_n \in G$. Using the expression (4) of E_n together with (7) and (8) prove our first assertion.

(2) Using the first assertion, we have

$$\begin{aligned} \left| e^{-\Delta/2, y} E_n(x, y) \right| &\leq \sum_{0 \leq m \leq n/2} \frac{1}{2^m m!} |\Delta^{m, y} E_n(x, y)| \\ &\leq \sum_{0 \leq m \leq n/2} \frac{d^m}{2^m m!} \frac{\|y\|^{n-2m}}{(n-2m)!} (\delta |G| \|x\|)^n. \end{aligned}$$

By consequence,

$$\begin{aligned} \sum_{n=0}^{\infty} \left| e^{-\Delta/2, y} E_n(x, y) \right| &\leq \sum_{n=0}^{\infty} (\delta |G| \|x\|)^n \sum_{0 \leq m \leq n/2} \frac{d^m}{2^m m!} \frac{\|y\|^{n-2m}}{(n-2m)!} \\ &\leq \sum_{m=0}^{\infty} \frac{d^m}{2^m m!} \sum_{n=2m}^{\infty} (\delta |G| \|x\|)^n \frac{\|y\|^{n-2m}}{(n-2m)!} \\ &\leq \left(\sum_{m=0}^{\infty} \frac{(\delta \sqrt{d} |G| \|x\|)^{2m}}{2^m m!} \right) \left(\sum_{n=0}^{\infty} (\delta |G| \|x\|)^n \frac{\|y\|^n}{n!} \right). \end{aligned}$$

This shows that the series converges uniformly with respect to x and y in each bounded subset of $\mathbb{R}^d \times \mathbb{R}^d$ and

$$\sum_{n=0}^{\infty} \left| e^{-\Delta/2, y} E_n(x, y) \right| \leq e^{(\delta \sqrt{d} |G| \|x\|)^2/2} e^{\delta |G| \|x\| \|y\|}$$

which proves our second assertion and ends the proof. \square

3. AN INTEGRAL REPRESENTATION FOR THE OPERATOR $V_k \circ e^{\Delta/2}$

Based on the results of the previous section, we will establish an integral representation for the operator $p \mapsto (V_k(e^{\Delta/2} p))(x)$ on the space of polynomials by means of a family of measures μ_x , $x \in \mathbb{R}^d$, which are absolutely continuous with respect to the Lebesgue measures in \mathbb{R}^d . These measures have the form

$$\mu_x(dz) := e^{-\|z\|^2/2} L_k(x, z) dz,$$

where L_k is a given kernel in $\mathbb{R}^d \times \mathbb{R}^d$. This section is mainly devoted to the study of the measures μ_x and the kernel L_k .

To do so, we need first to introduce some additional specific notations for this section. From now on, $d\gamma$ stands for the measure in \mathbb{R}^d given by

$$d\gamma := c_0^{-1} e^{-\|z\|^2/2} dz,$$

where dz is the Lebesgue measure in \mathbb{R}^d and $c_0 = (2\pi)^{d/2}$.

We let $L^p(\mathbb{R}^d)$ be the set of (classes of) functions defined in \mathbb{R}^d such that

$$\|f\|_{L^p(\mathbb{R}^d)}^p := \int_{\mathbb{R}^d} |f(z)|^p dz < +\infty,$$

and $L^2(\mathbb{R}^d, d\gamma)$ will consists of (classes of) of functions f defined in \mathbb{R}^d and satisfying

$$\|f\|_{2,\gamma}^2 := \int_{\mathbb{R}^d} |f(z)|^2 d\gamma(z) < +\infty.$$

For $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f will be defined by

$$\mathcal{F}(f)(y) := c_0^{-1} \int_{\mathbb{R}^d} f(z) e^{-i\langle y, z \rangle} dz.$$

Finally, using the standard multi-index notation in \mathbb{N}^d , we introduce the polynomials φ_ν , $\nu \in \mathbb{N}^d$, by:

$$\varphi_\nu(X) = \frac{1}{(\nu!)^{1/2}} X^\nu. \quad (9)$$

For a polynomial p , we denote by $p(\partial)$ the differential operator consisting by replacing X_j in p by $\partial_j := \frac{\partial}{\partial x_j}$, and we define the Fischer bilinear form $[\cdot, \cdot]$ in \mathcal{P} by

$$[p, q] := p(\partial)(q)(0) \quad (10)$$

The next Proposition gathers some well-known properties of this bilinear form.

Proposition 3.1. (1) *The system $(\varphi_\nu)_{\nu \in \mathbb{N}^d}$ forms an orthonormal basis of \mathcal{P} relatively to the Fischer bilinear form $[\cdot, \cdot]$.*

(2) *$[1, p] = p(0)$ for all polynomial p .*

(3) *If $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$ with $n \neq m$, then $[p, q] = 0$,*

(4) *$[x_i p, q] = [p, \partial_i q]$ for all $p, q \in \Pi$ and $i = 1, \dots, d$.*

Macdonald [8] states that

$$[p, q] = \int_{\mathbb{R}^d} \left(e^{-\Delta/2} p \right) (z) \left(e^{-\Delta/2} q \right) (z) d\gamma \quad (11)$$

for all polynomials p, q .

For $\nu \in \mathbb{N}^d$, we introduce the Hermite polynomial H_ν and its associated Hermite function h_ν by:

$$H_\nu(z) := \left(e^{-\Delta/2} \varphi_\nu \right) (z), \quad h_\nu := e^{-\|\cdot\|^2/2} H_\nu. \quad (12)$$

The systems $(H_\nu)_{\nu \in \mathbb{N}^d}$ and $(h_\nu)_{\nu \in \mathbb{N}^d}$ form an orthonormal bases of $L^2(\mathbb{R}^d, d\gamma)$ and $L^2(\mathbb{R}^d, c_0^{-1} dz)$ respectively, equipped with their standard inner product.

By use of Proposition 3.1.(4), and the multinomial formula

$$\frac{1}{n!} (X_1 + X_2 + \dots + X_d)^n = \sum_{\mu \in \mathbb{N}^d, |\mu|=n} \frac{X^\mu}{\mu!},$$

one has for all $\nu \in \mathbb{N}^d$, with $|\nu| = n$, the useful identity

$$\left[\langle x, \cdot \rangle^n, \varphi_\nu \right] = n! \varphi_\nu(x), \quad (13)$$

for all $x \in \mathbb{R}^d$, where the pairing $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^d .

The starting point of our integral representation is the next Proposition.

Proposition 3.2. *Let p be a polynomial. Then for all $x \in \mathbb{R}^d$ we have:*

$$V_k(p)(x) = \sum_{n=0}^{\infty} [E_n(x, \cdot), p].$$

Proof. Fix an integer $n \geq 0$. By (13), we see that

$$\frac{1}{n!} \langle x, y \rangle^n = \sum_{\nu \in \mathbb{N}^d, |\nu|=n} \varphi_\nu(x) \varphi_\nu(y), \quad (14)$$

for all $x, y \in \mathbb{R}^d$. Applying V_k , with respect to x , to both sides of (14) leads to

$$E_n(x, y) = \sum_{\nu \in \mathbb{N}^d, |\nu|=n} V_k(\varphi_\nu)(x) \varphi_\nu(y),$$

which in turn gives

$$V_k(\varphi_\nu)(x) = [E_n(x, \cdot), \varphi_\nu] \quad (15)$$

for all $\nu \in \mathbb{N}^d$ with $|\nu| = n$.

Fix $\nu \in \mathbb{N}^d$. By virtue of (15) and the orthogonality property of the Fisher pairing $[\cdot, \cdot]$ ensured by Proposition 3.1.(3), we may write

$$V_k(\varphi_\nu)(x) = \sum_{m=0}^{\infty} [E_m(x, \cdot), \varphi_\nu].$$

Since $(\varphi_\nu)_{\nu \in \mathbb{N}}$ is a basis of \mathcal{P} and V_k is linear, the latter formula holds for all polynomial p , and this proves our claim. \square

Our first main result is the following.

Theorem 3.3. *There exists a unique kernel $L_k(\cdot, \cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying*

$$V_k(p)(x) = \int_{\mathbb{R}^d} L_k(x, y) e^{-\Delta/2} p(y) d\gamma(y),$$

for all polynomial $p \in \mathcal{P}$ and all $x \in \mathbb{R}^d$. Moreover,

$$L_k(x, y) = \sum_{n=0}^{\infty} \left(e^{-\Delta/2, \cdot} E_n(x, \cdot) \right) (y), \quad (16)$$

and satisfies

$$|L_k(x, y)| \leq e^{(\delta\sqrt{d}|G|||x||)^2/2} e^{(\delta|G|||x|||y||)/2} \quad (17)$$

for all $x, y \in \mathbb{R}^d$.

Proof. Pick $x \in \mathbb{R}^d$ and fix $p \in \mathcal{P}$. From Proposition 3.2 we have

$$V_k(p)(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \left(e^{-\Delta/2, \cdot} E_n(x, \cdot) \right) (y) e^{-\Delta/2} p(y) d\gamma(y).$$

The dominate convergence theorem, thanks to Proposition 2.2, yields that

$$V_k(p)(x) = \int_{\mathbb{R}^d} L_k(x, y) e^{-\Delta/2} p(y) d\gamma(y),$$

where the kernel L_k is given by:

$$L_k(x, y) := \sum_{n=0}^{\infty} (e^{-\Delta/2, \cdot} E_n(x, \cdot))(y).$$

The unicity of L_k follows from the density of \mathcal{P} in $L^2(\mathbb{R}^d, d\gamma)$. The remaining is a direct application of Proposition 2.2, which completes the proof. \square

We begin by giving some direct properties of the kernel L_k . Others properties of L_k will be given bellow in their appropriate places.

Proposition 3.4. Fix $x \in \mathbb{R}^d$. We have:

- (1) $\int_{\mathbb{R}^d} L_k(x, y) d\gamma(y) = 1$,
- (2) L_k is continuous in $\mathbb{R}^d \times \mathbb{R}^d$,
- (3) L_k is real valued in $\mathbb{R}^d \times \mathbb{R}^d$ whenever k is real valued.
- (4) For $y \in \mathbb{R}^d$ fixed, the map $x \mapsto L_k(x, y)$ extends to a holomorphic function in \mathbb{C}^d .

Proof. (1) Follows from $V_k(1) = 1$.

(2) Follows from Proposition 2.2.

(3) Keeping remark 1.2 in mind, we see that $E_n(x, \cdot)$ is real valued when k is real valued, by consequence L_k is real valued in $\mathbb{R}^d \times \mathbb{R}^d$ when k is real valued.

(4) For y fixed, the maps $x \mapsto e^{-\Delta/2} E_n(x, y)$ are analytic in \mathbb{C}^d and arguing as in Proposition 2.2, we see that the series defining L_k converges uniformly with respect to $x \in \mathbb{C}^d$ in each bounded subset of \mathbb{C}^d , this ensures that $x \mapsto L_k(x, y)$ is holomorphic in \mathbb{C}^d .

Thus, all our claims are proved. \square

The next Proposition gives a generating series for the intertwining operator V_k .

Proposition 3.5. For all $x \in \mathbb{C}^d$, we have

$$L_k(x, \cdot) = \sum_{\nu \in \mathbb{N}^d} V_k(\varphi_\nu)(x) H_\nu(\cdot), \quad (18)$$

where the above equality holds point wise in \mathbb{R}^d and in $L^2(\mathbb{R}^d, d\gamma)$ as well. Moreover,

$$\left\| e^{-\|\cdot\|^2/4} L_k(x, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 = c_0^{-1} \sum_{\nu \in \mathbb{N}^d} \left| V_k(\varphi_\nu)(x) \right|^2 \quad (19)$$

Proof. Fix $x \in \mathbb{C}^d$. Theorem 3.3 gives us

$$V_k(\varphi_\nu)(x) = \int_{\mathbb{R}^d} L_k(x, y) (e^{-\Delta/2} \varphi_\nu)(y) d\gamma = \int_{\mathbb{R}^d} L_k(x, y) H_\nu(y) d\gamma.$$

Since H_ν is real valued and $(H_\nu)_{\nu \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d, d\gamma)$, we infer that

$$L_k(x, \cdot) = \sum_{\nu \in \mathbb{N}^d} V_k(\varphi_\nu)(x) H_\nu, \quad \text{in } L^2(\mathbb{R}^d, d\gamma).$$

This proves (18) in $L^2(\mathbb{R}^d, d\gamma)$.

Alternatively, by view of Lemma 2.1 and the estimates provided by Proposition 1.3, the series $y \mapsto \sum_{\nu \in \mathbb{N}^d} V_k(\varphi_\nu)(x) H_\nu(y)$ converges point wise in \mathbb{R}^d and uniformly in bounded subset of \mathbb{R}^d , and then (18) holds point wise almost every where w.r.t the lebesgue measure in \mathbb{R}^d . By a continuity argument, (18) holds actually every where in \mathbb{R}^d .

In other hand, (18) leads also to

$$e^{-\|\cdot\|^2/2} L_k(x, \cdot) = \sum_{\nu \in \mathbb{N}^d} V_k(\varphi_\nu)(x) h_\nu(\cdot), \quad \text{in } L^2(\mathbb{R}^2, c_0^{-1} dz)$$

and then taking the norm in $L^2(\mathbb{R}^d)$ in (18) we get

$$\left\| e^{-\|\cdot\|^2/4} L_k(x, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 = c_0^{-1} \sum_{\nu \in \mathbb{N}^d} \left| V_k(\varphi_\nu)(x) \right|^2,$$

which is the desired result. \square

Another important property of the kernel L_k is its relationship with the Dunkl kernel. The next theorem will show that the Dunkl kernel E_k is obtained as the convolution of the kernel L_k and the gaussian function $e^{-\|\cdot\|^2/2}$. This result will be the source of our second main result in Theorem 3.10 bellow.

Theorem 3.6. *For all $x, y \in \mathbb{C}^d$ we have*

$$E_k(x, y) = c_0^{-1} \int_{\mathbb{R}^d} L_k(x, z) e^{-\|y-z\|^2/2} dz = c_0^{-1} L_k(x, \cdot) * e^{-\|\cdot\|^2/2}(y)$$

Proof. Let us assume first that $x, y \in \mathbb{R}^d$. Note that in this case, Theorem 3.3 asserts that the integral is well defined for all $x, y \in \mathbb{R}^d$. In other hand, by the definition of L_k and Proposition 2.2 we have

$$\begin{aligned} \int_{\mathbb{R}^d} L_k(x, z) e^{-\|z-y\|^2/2} dz &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \left(e^{-\Delta/2, \cdot} E_n(x, \cdot) \right) (z) e^{-\|z-y\|^2/2} dz \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \left(e^{-\Delta/2, \cdot} E_n(x, \cdot) \right) (z+y) e^{-\|z\|^2/2} dz \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \left(e^{-\Delta/2, \cdot} E_n(x, \cdot+y) \right) (z) e^{-\|z\|^2/2} dz \\ &= c_0 \sum_{n=0}^{\infty} [E_n(x, \cdot+y), 1] \end{aligned}$$

From Proposition 10.(2) we have $E_n(x, y) = [E_n(x, \cdot+y), 1]$, from which the result for $x, y \in \mathbb{R}^d$ follows. The extension to $x, y \in \mathbb{C}^d$ is obtained by analytic continuation by view of the estimates on the kernel L_k furnished by Theorem 3.3. \square

A direct application is the following.

Corollary 3.7. *For all $x \in \mathbb{C}^d$, $y \in \mathbb{R}^d$ we have*

$$e^{-\|y\|^2/2} L_k(x, y) = \mathcal{F}(e^{-\|\cdot\|^2/2} E_k(-ix, \cdot))(y)$$

Proof. Noting that $E_k(ix, y) = E_k(x, iy)$, we deduce from Theorem 3.6 that for all $x, y \in \mathbb{R}^d$ we have

$$E_k(ix, y) = e^{\|y\|^2/2} c_0^{-1} \int_{\mathbb{R}^d} L_k(x, z) e^{-i\langle y, z \rangle} e^{-\|z\|^2/2} dz.$$

That is,

$$e^{-\|y\|^2/2} E_k(ix, y) = \mathcal{F}(e^{-\|\cdot\|^2/2} L_k(x, \cdot))(y)$$

for all $x, y \in \mathbb{R}^d$, and then

$$L_k(x, y) e^{-\|y\|^2/2} = \mathcal{F}(e^{-\|\cdot\|^2/2} E_k(-ix, \cdot))(y),$$

as claimed. \square

Proposition 3.8 and Proposition 3.9 bellow, are direct applications of Corollary 3.7 which reveal further properties of the kernel L_k .

Proposition 3.8. *We have $L_k \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and*

$$\partial_{y_j} \left(\left(L_k(x, \cdot) e^{-\|\cdot\|^2/2} \right) \right) (y) = T_j^x \left(\left(L_k(\cdot, y) e^{-\|y\|^2/2} \right) \right) (x),$$

for all $x, y \in \mathbb{R}^d$ and all $j = 1, \dots, d$.

Proposition 3.9. *For all $x, y \in \mathbb{R}^d$ and all $g \in G$, we have*

$$L_k(-x, y) = L_k(x, -y) \quad \text{and} \quad L_k(gx, y) = L_k(x, g^{-1}y),$$

One more fundamental result on the kernel L_k . It provides an interesting relationship between the kernel L_k and the intertwining operator V_k . This relationship reveals in particular, that the kernel L_k is non negative in the case where the parameter function k is non negative.

Theorem 3.10. *Assume that $\text{Re}(k) \geq 0$. Then for all $x, y \in \mathbb{R}^d$, we have*

$$V_k(e^{-\|\cdot+y\|^2/2})(x) = e^{-\|y\|^2/2} L_k(x, y).$$

As a consequence, the kernel L_k is non negative in $\mathbb{R}^d \times \mathbb{R}^d$ if k is non negative.

Proof. Fix $x \in \mathbb{R}^d$. Following dejeu [2, Theorem 5.1], we have

$$\begin{aligned} V_k(e^{-\|\cdot+y\|^2/2})(x) &= c_0^{-1} \int_{\mathbb{R}^d} \mathcal{F}(e^{-\|\cdot+y\|^2/2})(z) E_k(ix, z) dz \\ &= c_0^{-1} \int_{\mathbb{R}^d} \mathcal{F}(e^{-\|\cdot\|^2/2})(z) e^{i\langle y, z \rangle} E_k(ix, z) dz \\ &= c_0^{-1} \int_{\mathbb{R}^d} e^{-\|z\|^2/2} e^{i\langle y, z \rangle} E_k(ix, z) dz \end{aligned}$$

whence,

$$V_k(e^{-\|\cdot+y\|^2/2})(x) = \mathcal{F} \left(e^{-\|\cdot\|^2/2} E_k(-ix, \cdot) \right) (y),$$

and our first claim follows from Corollary 3.7.

Assume now that k is non negative. Following Rösler [12], $V_k(e^{-\|\cdot+y\|^2/2})(x) \geq 0$ for all $x, y \in \mathbb{R}^d$, we infer that L_k is non negative in $\mathbb{R}^d \times \mathbb{R}^d$ and this ends the proof. \square

Our second main result in this paper is about the extension of $V_k \circ e^{\Delta/2}$ as a bounded operator to a large class of functions than polynomials.

A linear form Λ defined on some functional space E will be called positive or positivity-preserving, if we have:

$$f \in E, f \geq 0 \implies \Lambda(f) \geq 0.$$

Theorem 3.11. *Let $x \in \mathbb{R}^d$. Then the linear functional*

$$\Phi_{x,k} := \Phi_x : p \mapsto V_k(e^{\Delta/2}p)(x)$$

defined on polynomials, extends uniquely to $L^2(\mathbb{R}^d, d\gamma)$ as a bounded operator, bearing the same name, with:

$$\|\Phi_x\| = c_0^{-1/2} \left(\sum_{\nu \in \mathbb{N}^d} |V_k(\varphi_\nu)(x)|^2 \right)^{1/2}.$$

Further, in the case where k is non negative, Φ_x is positive on $L^2(\mathbb{R}^d, d\gamma)$.

Proof. Appealing to Theorem 3.3 we see, using Hölder inequality, that

$$\left| V_k(e^{\Delta/2}p)(x) \right| \leq \|L_k(x, \cdot)\|_{2,\gamma} \|p\|_{2,\gamma}$$

for all polynomial p . Since polynomials are dense in $L^2(\mathbb{R}^d, d\gamma)$, then Φ_x extends uniquely, conserving the same name, to the space $f \in L^2(\mathbb{R}^d, d\gamma)$ by setting

$$\Phi_x(f) := \int_{\mathbb{R}^d} L_k(x, y) f(y) d\gamma(y)$$

for all $f \in L^2(\mathbb{R}^d, d\gamma)$. Hence $\|\Phi_x\| = \|L_k(x, \cdot)\|_{L^2(\mathbb{R}^d, d\gamma)}$ and by view of Proposition 3.5, our first claim follows.

If further k is non negative, then by Theorem 3.3, the kernel L_k is non negative in $\mathbb{R}^d \times \mathbb{R}^d$, and this completes the proof. \square

We conclude this work by noting that Theorem 3.11 allows us to define $e^{\Delta/2}(f)$ for $f \in L^2(\mathbb{R}^d, d\gamma)$ by setting

$$(e^{\Delta/2}f)(z) := V_k^{-1} \left[\left(V_k \circ e^{\Delta/2} \right) (f) \right] (z), \quad z \in \mathbb{R}^d.$$

Indeed: The right hand side of the above formula makes sense since by application of Theorem 3.11, we see that for each $f \in L^2(\mathbb{R}^d, d\gamma)$ the function

$$\psi : x \mapsto \Phi_x(f) = \left(V_k \circ e^{\Delta/2} \right) (f)(x)$$

is smooth in \mathbb{R}^d , and then $V_k^{-1}(\psi)$ is well defined.

REFERENCES

- [1] M. F. E. de Jeu, The Dunkl transform, *Invent. Math.*, **113** (1993), 147-162
- [2] M.F.E de jeu, Paley-Wiener theorems for the Dunkl transform, *Trans. Amer. Math. Soc.*, **10**, **358**, 2006, 4225-4250.
- [3] C. F. Dunkl, Integral kernels with reflection group invariance, *Can. J. Math.*, **43** 1991, 1213-1227.
- [4] C. F. Dunkl, Hankel transforms associated to finite reflection groups. Hypergeometric functions on domains of positivity, Jack polynomials, and applications. *Contemp. Math.* **138** 1992, 123-138.
- [5] C. F. Dunkl and Yuan Xu, Orthogonal Polynomials of Several Variables, Cambridge University Press, 2001.
- [6] C. F. Dunkl, M. F. E. de Jeu and E. Opdam, Singular polynomials for finite reflection group, *Trans. Amer. Math. Soc.*, **346**, 1994, 237-256.
- [7] L. Lapointe et L. Vinet, Exact operator solution of the Calogero-Sutherland model, *J. Phys.*, 425-452, **178**, 1996.
- [8] Macdonald, I.G., The Volume of a Compact Lie Group. *Invent. Math.* **56** , 1980, 93-95.
- [9] M. Maslouhi and E. Youssfi. The Dunkl intertwining operator. *JFA*, **256** (8) 2009, 2697-2709.
- [10] M. Maslouhi and E. Youssfi. Corrigendum to "The Dunkl intertwining operator". *JFA*. **258** 2010, 2862-2864.
- [11] L. A. Harris, Bernstein's polynomial inequalities and functional analysis, *Irish Math. Soc. Bull.* **36**. 1996, 19-33.
- [12] M. Rösler. Positivity of Dunkl's intertwining operator. *Duke Math. J.*, **98**, 1999, 445-463.
- [13] M. Rosler and M. Voit, Markov processes related with Dunkl Operators *Adv. Appl. Math.*, **21**, 1998, 575-643.

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